5.5 Bases Other than *e* and Applications

- **Define exponential functions that have bases other than e.**
- **Differentiate and integrate exponential functions that have bases other than e.**
- **Use exponential functions to model compound interest and exponential growth.**

Bases Other than e

The **base** of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

> **Definition of Exponential Function to Base *a***
>
> If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base *a*** is denoted by a^x and is defined by
>
> $$a^x = e^{(\ln a)x}.$$
>
> If $a = 1$, then $y = 1^x = 1$ is a constant function.

These functions obey the usual laws of exponents. For instance, here are some familiar properties.

1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$ **3.** $\dfrac{a^x}{a^y} = a^{x-y}$ **4.** $(a^x)^y = a^{xy}$

When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model. (*Half-life* is the number of years required for half of the atoms in a sample of radioactive material to decay.)

EXAMPLE 1 Radioactive Half-Life Model

The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbon-14. How much will be present in 10,000 years?

Carbon dating uses the radioisotope carbon-14 to estimate the age of dead organic materials. The method is based on the decay rate of carbon-14 (see Example 1), a compound organisms take in when they are alive.

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Solution Let $t = 0$ represent the present time and let y represent the amount (in grams) of carbon-14 in the sample. Using a base of $\frac{1}{2}$, you can model y by the equation

$$y = \left(\frac{1}{2}\right)^{t/5715}.$$

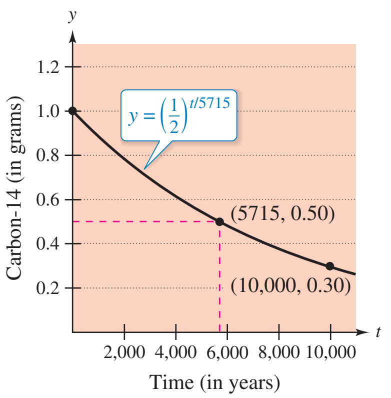
Notice that when $t = 5715$, the amount is reduced to half of the original amount.

$$y = \left(\frac{1}{2}\right)^{5715/5715} = \frac{1}{2} \text{ gram}$$

When $t = 11{,}430$, the amount is reduced to a quarter of the original amount, and so on. To find the amount of carbon-14 after 10,000 years, substitute 10,000 for t.

$$y = \left(\frac{1}{2}\right)^{10{,}000/5715}$$
$$\approx 0.30 \text{ gram}$$

The graph of y is shown in Figure 5.23.



The half-life of carbon-14 is about 5715 years.

Figure 5.23

Logarithmic functions to bases other than e can be defined in much the same way as exponential functions to other bases are defined.

REMARK In precalculus, you learned that $\log_a x$ is the value to which a must be raised to produce x. This agrees with the definition at the right because

$$\begin{aligned} a^{\log_a x} &= a^{(1/\ln a)\ln x} \\ &= (e^{\ln a})^{(1/\ln a)\ln x} \\ &= e^{(\ln a/\ln a)\ln x} \\ &= e^{\ln x} \\ &= x. \end{aligned}$$

Definition of Logarithmic Function to Base a

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Logarithmic functions to the base a have properties similar to those of the natural logarithmic function given in Theorem 5.2. (Assume x and y are positive numbers and n is rational.)

1. $\log_a 1 = 0$ — Log of 1
2. $\log_a xy = \log_a x + \log_a y$ — Log of a product
3. $\log_a x^n = n \log_a x$ — Log of a power
4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ — Log of a quotient

From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$, for all x

The logarithmic function to the base 10 is called the **common logarithmic function.** So, for common logarithms,

$$y = 10^x \quad \text{if and only if} \quad x = \log_{10} y. \qquad \text{Property of Inverse Functions}$$

EXAMPLE 2 **Bases Other than e**

Solve for x in each equation.

a. $3^x = \dfrac{1}{81}$ **b.** $\log_2 x = -4$

Solution

a. To solve this equation, you can apply the logarithmic function to the base 3 to each side of the equation.

$$\begin{aligned} 3^x &= \frac{1}{81} \\ \log_3 3^x &= \log_3 \frac{1}{81} \\ x &= \log_3 3^{-4} \\ x &= -4 \end{aligned}$$

b. To solve this equation, you can apply the exponential function to the base 2 to each side of the equation.

$$\begin{aligned} \log_2 x &= -4 \\ 2^{\log_2 x} &= 2^{-4} \\ x &= \frac{1}{2^4} \\ x &= \frac{1}{16} \end{aligned}$$

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the differentiation rules for bases other than e given in the next theorem.

REMARK These differentiation rules are similar to those for the natural exponential function and the natural logarithmic function. In fact, they differ only by the constant factors $\ln a$ and $1/\ln a$. This points out one reason why, for calculus, e is the most convenient base.

THEOREM 5.13 Derivatives for Bases Other than e

Let a be a positive real number $(a \neq 1)$, and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$

2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u \dfrac{du}{dx}$

3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$

4. $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u}\dfrac{du}{dx}$

Proof By definition, $a^x = e^{(\ln a)x}$. So, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u \frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

To prove the third rule, you can write

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a}\ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

The second and fourth rules are simply the Chain Rule versions of the first and third rules.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 3 Differentiating Functions to Other Bases

Find the derivative of each function.

a. $y = 2^x$ **b.** $y = 2^{3x}$ **c.** $y = \log_{10} \cos x$ **d.** $y = \log_3 \dfrac{\sqrt{x}}{x+5}$

Solution

a. $y' = \dfrac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \dfrac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3 \ln 2)2^{3x}$

REMARK Try writing 2^{3x} as 8^x and differentiating to see that you obtain the same result.

c. $y' = \dfrac{d}{dx}[\log_{10} \cos x] = \dfrac{-\sin x}{(\ln 10)\cos x} = -\dfrac{1}{\ln 10}\tan x$

d. Before differentiating, rewrite the function using logarithmic properties.

$$y = \log_3 \frac{\sqrt{x}}{x+5} = \frac{1}{2}\log_3 x - \log_3(x+5)$$

Next, apply Theorem 5.13 to differentiate the function.

$$\begin{aligned} y' &= \frac{d}{dx}\left[\frac{1}{2}\log_3 x - \log_3(x+5)\right] \\ &= \frac{1}{2(\ln 3)x} - \frac{1}{(\ln 3)(x+5)} \\ &= \frac{5-x}{2(\ln 3)x(x+5)} \end{aligned}$$

Occasionally, an integrand involves an exponential function to a base other than e. When this occurs, there are two options: (1) convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or (2) integrate directly, using the integration formula

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$$

which follows from Theorem 5.13.

EXAMPLE 4 **Integrating an Exponential Function to Another Base**

Find $\displaystyle\int 2^x\,dx$.

Solution

$$\int 2^x\,dx = \frac{1}{\ln 2}2^x + C$$

When the Power Rule, $D_x[x^n] = nx^{n-1}$, was introduced in Chapter 2, the exponent n was required to be a rational number. Now the rule is extended to cover any real value of n. Try to prove this theorem using logarithmic differentiation.

THEOREM 5.14 The Power Rule for Real Exponents

Let n be any number, and let u be a differentiable function of x.

1. $\displaystyle\frac{d}{dx}[x^n] = nx^{n-1}$ **2.** $\displaystyle\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$

The next example compares the derivatives of four types of functions. Each function uses a different differentiation formula, depending on whether the base and the exponent are constants or variables.

EXAMPLE 5 **Comparing Variables and Constants**

a. $\displaystyle\frac{d}{dx}[e^e] = 0$ Constant Rule

b. $\displaystyle\frac{d}{dx}[e^x] = e^x$ Exponential Rule

c. $\displaystyle\frac{d}{dx}[x^e] = ex^{e-1}$ Power Rule

d. $y = x^x$ Logarithmic differentiation

$$\ln y = \ln x^x$$
$$\ln y = x\ln x$$
$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1)$$
$$\frac{y'}{y} = 1 + \ln x$$
$$y' = y(1 + \ln x)$$
$$y' = x^x(1 + \ln x)$$

REMARK Be sure you see that there is no simple differentiation rule for calculating the derivative of $y = x^x$. In general, when $y = u(x)^{v(x)}$, you need to use logarithmic differentiation.

Applications of Exponential Functions

An amount of P dollars is deposited in an account at an annual interest rate r (in decimal form). What is the balance in the account at the end of 1 year? The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^n.$$

n	A
1	\$1080.00
2	\$1081.60
4	\$1082.43
12	\$1083.00
365	\$1083.28

For instance, the result for a deposit of \$1000 at 8% interest compounded n times a year is shown in the table at the right.

As n increases, the balance A approaches a limit. To develop this limit, use the next theorem. To test the reasonableness of this theorem, try evaluating

$$\left(\frac{x+1}{x}\right)^x$$

for several values of x, as shown in the table at the left.

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

THEOREM 5.15 A Limit Involving *e*

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Given Theorem 5.15, take another look at the formula for the balance A in an account in which the interest is compounded n times per year. By taking the limit as n approaches infinity, you obtain

$$\begin{aligned}
A &= \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^n && \text{Take limit as } n \to \infty. \\
&= P \lim_{n\to\infty}\left[\left(1 + \frac{1}{n/r}\right)^{n/r}\right]^r && \text{Rewrite.} \\
&= P\left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right]^r && \text{Let } x = n/r. \text{ Then } x \to \infty \text{ as } n \to \infty. \\
&= Pe^r. && \text{Apply Theorem 5.15.}
\end{aligned}$$

This limit produces the balance after 1 year of **continuous compounding.** So, for a deposit of \$1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$$A = 1000e^{0.08} \approx \$1083.29.$$

SUMMARY OF COMPOUND INTEREST FORMULAS

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (decimal form), and n = number of compoundings per year.

1. Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. Compounded continuously: $A = Pe^{rt}$

EXAMPLE 6 Continuous, Quarterly, and Monthly Compounding

See LarsonCalculus.com for an interactive version of this type of example.

A deposit of \$2500 is made in an account that pays an annual interest rate of 5%. Find the balance in the account at the end of 5 years when the interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

Solution

a. $$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Compounded quarterly} \\ &= 2500\left(1 + \frac{0.05}{4}\right)^{4(5)} \\ &= 2500(1.0125)^{20} \\ &\approx \$3205.09 \end{aligned}$$

b. $$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Compounded monthly} \\ &= 2500\left(1 + \frac{0.05}{12}\right)^{12(5)} \\ &\approx 2500(1.0041667)^{60} \\ &\approx \$3208.40 \end{aligned}$$

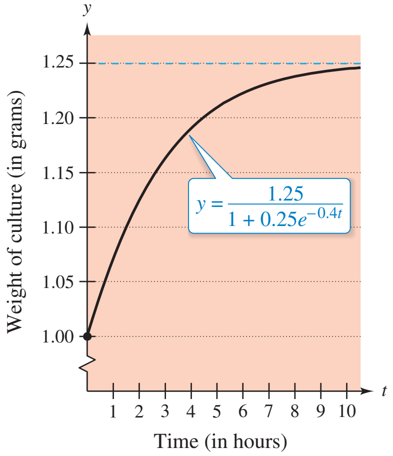
c. $$\begin{aligned} A &= Pe^{rt} && \text{Compounded continuously} \\ &= 2500\left[e^{0.05(5)}\right] \\ &= 2500e^{0.25} \\ &\approx \$3210.06 \end{aligned}$$

EXAMPLE 7 Bacterial Culture Growth

A bacterial culture is growing according to the *logistic growth function*

$$y = \frac{1.25}{1 + 0.25e^{-0.4t}}, \quad t \geq 0$$

where y is the weight of the culture in grams and t is the time in hours. Find the weight of the culture after (a) 0 hours, (b) 1 hour, and (c) 10 hours. (d) What is the limit as t approaches infinity?



The limit of the weight of the culture as $t \to \infty$ is 1.25 grams.

Figure 5.24

Solution

a. When $t = 0$, $$y = \frac{1.25}{1 + 0.25e^{-0.4(0)}} = 1 \text{ gram.}$$

b. When $t = 1$, $$y = \frac{1.25}{1 + 0.25e^{-0.4(1)}} \approx 1.071 \text{ grams.}$$

c. When $t = 10$, $$y = \frac{1.25}{1 + 0.25e^{-0.4(10)}} \approx 1.244 \text{ grams.}$$

d. Taking the limit as t approaches infinity, you obtain

$$\lim_{t \to \infty} \frac{1.25}{1 + 0.25e^{-0.4t}} = \frac{1.25}{1 + 0} = 1.25 \text{ grams.}$$

The graph of the function is shown in Figure 5.24.

5.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Logarithmic Expression In Exercises 1–4, evaluate the expression without using a calculator.

1. $\log_2 \frac{1}{8}$
2. $\log_{27} 9$
3. $\log_7 1$
4. $\log_a \dfrac{1}{a}$

Exponential and Logarithmic Forms of Equations In Exercises 5–8, write the exponential equation as a logarithmic equation or vice versa.

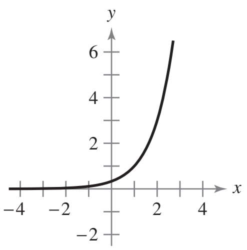
5. (a) $2^3 = 8$
   (b) $3^{-1} = \frac{1}{3}$
6. (a) $27^{2/3} = 9$
   (b) $16^{3/4} = 8$
7. (a) $\log_{10} 0.01 = -2$
   (b) $\log_{0.5} 8 = -3$
8. (a) $\log_3 \frac{1}{9} = -2$
   (b) $49^{1/2} = 7$

Sketching a Graph In Exercises 9–14, sketch the graph of the function by hand.

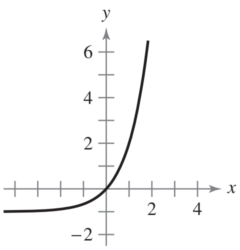
9. $y = 2^x$
10. $y = 4^{x-1}$
11. $y = \left(\frac{1}{3}\right)^x$
12. $y = 2^{x^2}$
13. $h(x) = 5^{x-2}$
14. $y = 3^{-|x|}$

Matching In Exercises 15–18, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

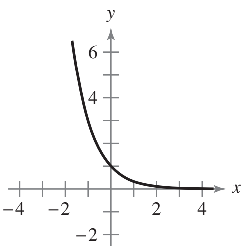
(a)



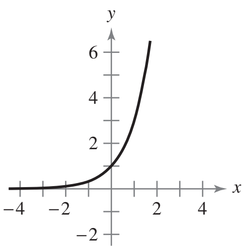
(b)



(c)



(d)



15. $f(x) = 3^x$
16. $f(x) = 3^{-x}$
17. $f(x) = 3^x - 1$
18. $f(x) = 3^{x-1}$

Solving an Equation In Exercises 19–24, solve for x or b.

19. (a) $\log_{10} 1000 = x$
    (b) $\log_{10} 0.1 = x$
20. (a) $\log_3 \frac{1}{81} = x$
    (b) $\log_6 36 = x$
21. (a) $\log_3 x = -1$
    (b) $\log_2 x = -4$
22. (a) $\log_b 27 = 3$
    (b) $\log_b 125 = 3$
23. (a) $x^2 - x = \log_5 25$
    (b) $3x + 5 = \log_2 64$
24. (a) $\log_3 x + \log_3(x - 2) = 1$
    (b) $\log_{10}(x + 3) - \log_{10} x = 1$

Solving an Equation In Exercises 25–34, solve the equation accurate to three decimal places.

25. $3^{2x} = 75$
26. $5^{6x} = 8320$
27. $2^{3-z} = 625$
28. $3(5^{x-1}) = 86$
29. $\left(1 + \dfrac{0.09}{12}\right)^{12t} = 3$
30. $\left(1 + \dfrac{0.10}{365}\right)^{365t} = 2$
31. $\log_2(x - 1) = 5$
32. $\log_{10}(t - 3) = 2.6$
33. $\log_3 x^2 = 4.5$
34. $\log_5 \sqrt{x - 4} = 3.2$

Verifying Inverse Functions In Exercises 35 and 36, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.

35. $f(x) = 4^x$
    $g(x) = \log_4 x$
36. $f(x) = 3^x$
    $g(x) = \log_3 x$

Finding a Derivative In Exercises 37–58, find the derivative of the function. (***Hint:*** In some exercises, you may find it helpful to apply logarithmic properties *before* differentiating.)

37. $f(x) = 4^x$
38. $f(x) = 3^{4x}$
39. $y = 5^{-4x}$
40. $y = 6^{3x-4}$
41. $f(x) = x\,9^x$
42. $y = x(6^{-2x})$
43. $g(t) = t^2 2^t$
44. $f(t) = \dfrac{3^{2t}}{t}$
45. $h(\theta) = 2^{-\theta} \cos \pi\theta$
46. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$
47. $y = \log_4(5x + 1)$
48. $y = \log_3(x^2 - 3x)$
49. $h(t) = \log_5(4 - t)^2$
50. $g(t) = \log_2(t^2 + 7)^3$
51. $y = \log_5 \sqrt{x^2 - 1}$
52. $f(x) = \log_2 \sqrt[3]{2x + 1}$
53. $f(x) = \log_2 \dfrac{x^2}{x - 1}$
54. $y = \log_{10} \dfrac{x^2 - 1}{x}$
55. $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$
56. $g(x) = \log_5 \dfrac{4}{x^2\sqrt{1 - x}}$
57. $g(t) = \dfrac{10 \log_4 t}{t}$
58. $f(t) = t^{3/2} \log_2 \sqrt{t + 1}$

Finding an Equation of a Tangent Line In Exercises 59–62, find an equation of the tangent line to the graph of the function at the given point.

59. $y = 2^{-x}, \quad (-1, 2)$
60. $y = 5^{x-2}, \quad (2, 1)$
61. $y = \log_3 x, \quad (27, 3)$
62. $y = \log_{10} 2x, \quad (5, 1)$

Logarithmic Differentiation In Exercises 63–66, use logarithmic differentiation to find dy/dx.

63. $y = x^{2/x}$
64. $y = x^{x-1}$
65. $y = (x - 2)^{x+1}$
66. $y = (1 + x)^{1/x}$

Finding an Equation of a Tangent Line **In Exercises 67–70, find an equation of the tangent line to the graph of the function at the given point.**

67. $y = x^{\sin x}, \quad \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

68. $y = (\sin x)^{2x}, \quad \left(\frac{\pi}{2}, 1\right)$

69. $y = (\ln x)^{\cos x}, \quad (e, 1)$

70. $y = x^{1/x}, \quad (1, 1)$

Finding an Indefinite Integral **In Exercises 71–78, find the indefinite integral.**

71. $\int 3^x \, dx$

72. $\int 8^{-x} \, dx$

73. $\int (x^2 + 2^{-x}) \, dx$

74. $\int (x^4 + 5^x) \, dx$

75. $\int x(5^{-x^2}) \, dx$

76. $\int (x + 4)6^{(x+4)^2} \, dx$

77. $\int \frac{3^{2x}}{1 + 3^{2x}} \, dx$

78. $\int 2^{\sin x} \cos x \, dx$

Evaluating a Definite Integral **In Exercises 79–82, evaluate the definite integral.**

79. $\int_{-1}^{2} 2^x \, dx$

80. $\int_{-4}^{4} 3^{x/4} \, dx$

81. $\int_{0}^{1} (5^x - 3^x) \, dx$

82. $\int_{1}^{3} (7^x - 4^x) \, dx$

Area **In Exercises 83 and 84, find the area of the region bounded by the graphs of the equations.**

83. $y = 3^x, y = 0, x = 0, x = 3$

84. $y = 3^{\cos x} \sin x, y = 0, x = 0, x = \pi$

WRITING ABOUT CONCEPTS

85. Analyzing a Logarithmic Equation Consider the function $f(x) = \log_{10} x$.

(a) What is the domain of f?

(b) Find f^{-1}.

(c) Let x be a real number between 1000 and 10,000. Determine the interval in which $f(x)$ will be found.

(d) Determine the interval in which x will be found if $f(x)$ is negative.

(e) When $f(x)$ is increased by one unit, x must have been increased by what factor?

(f) Find the ratio of x_1 to x_2 given that $f(x_1) = 3n$ and $f(x_2) = n$.

86. Comparing Rates of Growth Order the functions

$$f(x) = \log_2 x, \quad g(x) = x^x, \quad h(x) = x^2, \quad \text{and} \quad k(x) = 2^x$$

from the one with the greatest rate of growth to the one with the least rate of growth for large values of x.

87. Inflation When the annual rate of inflation averages 5% over the next 10 years, the approximate cost C of goods or services during any year in that decade is

$$C(t) = P(1.05)^t$$

where t is the time in years and P is the present cost.

(a) The price of an oil change for your car is presently \$24.95. Estimate the price 10 years from now.

(b) Find the rates of change of C with respect to t when $t = 1$ and $t = 8$.

(c) Verify that the rate of change of C is proportional to C. What is the constant of proportionality?

88. Depreciation After t years, the value of a car purchased for \$25,000 is

$$V(t) = 25{,}000\left(\tfrac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function and determine the value of the car 2 years after it was purchased.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to graph $V'(t)$ and determine the horizontal asymptote of $V'(t)$. Interpret its meaning in the context of the problem.

Compound Interest **In Exercises 89–92, complete the table by determining the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous Compounding
A						

89. $P = \$1000$
$r = 3\frac{1}{2}\%$
$t = 10$ years

90. $P = \$2500$
$r = 6\%$
$t = 20$ years

91. $P = \$1000$
$r = 5\%$
$t = 30$ years

92. $P = \$4000$
$r = 4\%$
$t = 15$ years

Compound Interest **In Exercises 93–96, complete the table by determining the amount of money P (present value) that should be invested at rate r to produce a balance of \$100,000 in t years.**

t	1	10	20	30	40	50
P						

93. $r = 5\%$
Compounded continuously

94. $r = 3\%$
Compounded continuously

95. $r = 5\%$
Compounded monthly

96. $r = 2\%$
Compounded daily

97. Compound Interest Assume that you can earn 6% on an investment, compounded daily. Which of the following options would yield the greatest balance after 8 years?

(a) \$20,000 now (b) \$30,000 after 8 years

(c) \$8000 now and \$20,000 after 4 years

(d) \$9000 now, \$9000 after 4 years, and \$9000 after 8 years

98. Compound Interest Consider a deposit of \$100 placed in an account for 20 years at r% compounded continuously. Use a graphing utility to graph the exponential functions describing the growth of the investment over the 20 years for the following interest rates. Compare the ending balances for the three rates.

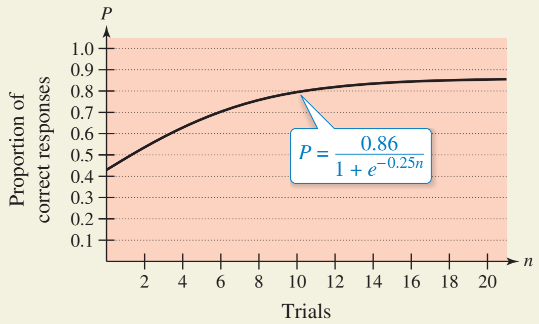
(a) $r = 3\%$ (b) $r = 5\%$ (c) $r = 6\%$

99. Timber Yield The yield V (in millions of cubic feet per acre) for a stand of timber at age t is $V = 6.7e^{(-48.1)/t}$, where t is measured in years.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rates at which the yield is changing when $t = 20$ years and $t = 60$ years.

100. HOW DO YOU SEE IT? The graph shows the proportion P of correct responses after n trials in a group project in learning theory.



(a) What is the limiting proportion of correct responses as n approaches infinity?

(b) What happens to the rate of change of the proportion in the long run?

101. Population Growth A lake is stocked with 500 fish, and the population increases according to the logistic curve

$$p(t) = \frac{10,000}{1 + 19e^{-t/5}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function.

(b) What is the limiting size of the fish population?

(c) At what rates is the fish population changing at the end of 1 month and at the end of 10 months?

(d) After how many months is the population increasing most rapidly?

102. Modeling Data The breaking strengths B (in tons) of steel cables of various diameters d (in inches) are shown in the table.

d	0.50	0.75	1.00	1.25	1.50	1.75
B	9.85	21.8	38.3	59.2	84.4	114.0

(a) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the rates of growth of the model when $d = 0.8$ and $d = 1.5$.

103. Comparing Models The numbers y of pancreas transplants in the United States for the years 2004 through 2010 are shown in the table, with $x = 4$ corresponding to 2004. *(Source: Organ Procurement and Transplantation Network)*

x	4	5	6	7	8	9	10
y	603	542	466	468	436	376	350

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$ $y_2 = a + b \ln x$

$y_3 = ab^x$ $y_4 = ax^b$

(b) Use a graphing utility to plot the data and graph each of the models. Which model do you think best fits the data?

(c) Interpret the slope of the linear model in the context of the problem.

(d) Find the rate of change of each of the models for the year 2008. Which model is decreasing at the greatest rate in 2008?

104. An Approximation of *e* Complete the table to demonstrate that e can also be defined as

$$\lim_{x \to 0^+} (1 + x)^{1/x}.$$

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

Modeling Data **In Exercises 105 and 106, find an exponential function that fits the experimental data collected over time t.**

105.

t	0	1	2	3	4
y	1200.00	720.00	432.00	259.20	155.52

106.

t	0	1	2	3	4
y	600.00	630.00	661.50	694.58	729.30

Using Properties of Exponents **In Exercises 107–110, find the exact value of the expression.**

107. $5^{1/\ln 5}$

108. $6^{\ln 10/\ln 6}$

109. $9^{1/\ln 3}$

110. $32^{1/\ln 2}$

True or False? **In Exercises 111–116, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

111. $e = \dfrac{271{,}801}{99{,}900}$

112. If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of n.

113. The functions $f(x) = 2 + e^x$ and $g(x) = \ln(x - 2)$ are inverse functions of each other.

114. The exponential function $y = Ce^x$ is a solution of the differential equation

$$\frac{d^n y}{dx^n} = y, \quad n = 1, 2, 3, \ldots.$$

115. The graphs of $f(x) = e^x$ and $g(x) = e^{-x}$ meet at right angles.

116. If $f(x) = g(x)e^x$, then the only zeros of f are the zeros of g.

117. Comparing Functions

(a) Show that $(2^3)^2 \neq 2^{(3^2)}$.

(b) Are $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$ the same function? Why or why not?

(c) Find $f'(x)$ and $g'(x)$.

118. Finding an Inverse Function Let

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for $a > 0$, $a \neq 1$. Show that f has an inverse function. Then find f^{-1}.

119. Logistic Differential Equation Show that solving the logistic differential equation

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

results in the logistic growth function in Example 7.

$$\left[\textit{Hint: } \frac{1}{y\left(\frac{5}{4} - y\right)} = \frac{4}{5}\left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right)\right]$$

120. Using Properties of Exponents Given the exponential function $f(x) = a^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$.

(b) $f(2x) = [f(x)]^2$.

121. Tangent Lines

(a) Determine y' given $y^x = x^y$.

(b) Find the slope of the tangent line to the graph of $y^x = x^y$ at each of the following points.

(i) (c, c) (ii) $(2, 4)$ (iii) $(4, 2)$

(c) At what points on the graph of $y^x = x^y$ does the tangent line not exist?

PUTNAM EXAM CHALLENGE

122. Which is greater

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} \quad \text{or} \quad \left(\sqrt{n+1}\right)^{\sqrt{n}}$$

where $n > 8$?

123. Show that if x is positive, then

$$\log_e\left(1 + \frac{1}{x}\right) > \frac{1}{1 + x}.$$

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SECTION PROJECT

Using Graphing Utilities to Estimate Slope

Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Use a graphing utility to graph f in the viewing window $-3 \le x \le 3$, $-2 \le y \le 2$. What is the domain of f?

(b) Use the *zoom* and *trace* features of a graphing utility to estimate

$$\lim_{x \to 0} f(x).$$

(c) Write a short paragraph explaining why the function f is continuous for all real numbers.

(d) Visually estimate the slope of f at the point $(0, 1)$.

(e) Explain why the derivative of a function can be approximated by the formula

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

for small values of Δx. Use this formula to approximate the slope of f at the point $(0, 1)$.

$$f'(0) \approx \frac{f(0 + \Delta x) - f(0 - \Delta x)}{2\Delta x} = \frac{f(\Delta x) - f(-\Delta x)}{2\Delta x}$$

What do you think the slope of the graph of f is at $(0, 1)$?

(f) Find a formula for the derivative of f and determine $f'(0)$. Write a short paragraph explaining how a graphing utility might lead you to approximate the slope of a graph incorrectly.

(g) Use your formula for the derivative of f to find the relative extrema of f. Verify your answer using a graphing utility.

FOR FURTHER INFORMATION For more information on using graphing utilities to estimate slope, see the article "Computer-Aided Delusions" by Richard L. Hall in *The College Mathematics Journal*. To view this article, go to *MathArticles.com.*